1. Representation of Curves and Surfaces

We first introduce three forms to represent geometric objects mathematically. They are the *parametric*, *implicit* and *explicit* forms. Implicit and explicit forms are often referred to as *nonparametric* forms. Then we briefly review the representation of curves and surfaces in Bézier and B-spline form and treat the special properties associated with each.

1.1 Analytic representation of curves

1.1.1 Plane curves

A plane curve can be expressed in the *parametric* form as

$$x = x(t), \quad y = y(t) \ , \tag{1.1}$$

where the coordinates of the point (x, y) of the curve are expressed as functions of a parameter t within a closed interval $t_1 \leq t \leq t_2$. The functions $x(t)$ and $y(t)$ are assumed to be continuous with a sufficient number of continuous derivatives. The parametric curve is said to be of *class* r, if the functions have continuous derivatives up to the order r, inclusively [206]. In vector notation the parametric curve can be specified by a vector-valued function

$$\mathbf{r} = \mathbf{r}(t) \ . \tag{1.2}$$

Another method of representing a curve analytically is to impose one condition on a variable point (x, y) by an equation of the form

$$f(x, y) = 0 \ . \tag{1.3}$$

This is an *implicit* equation for a plane curve. When $f(x, y)$ is linear in variables x and y, (1.3) represents a straight line. If $f(x, y)$ is of the second degree in x and y (i.e. $ax^2 + 2bxy + cy^2 + 2dx + 2ey + h = 0$), (1.3) represents a variety of plane curves called *conic* sections [79]. The implicit equation for a plane curve can also be expressed as an intersection curve between a parametric surface and a plane. We will discuss this formulation in Chap. 5.

The *explicit* form can be considered as a special case of parametric and implicit forms. If t can be expressed as a function of x or y, we can easily eliminate t from (1.1) to generate the explicit form

$$y = F(x) \quad or \quad x = G(y)\ . \tag{1.4}$$

This is always possible at least locally when $\frac{dx}{dt} \neq 0$ or $\frac{dy}{dt} \neq 0$ [412]. Conversely if we set x or y in (1.4) to be equal to the parameter t we obtain the parametric form (1.1). Also if the implicit equation (1.3) can be solved for one variable in terms of the other, we also obtain (1.4). This is always possible at least locally when $\frac{\partial f}{\partial y} \neq 0$ or $\frac{\partial f}{\partial x} \neq 0$ [166].

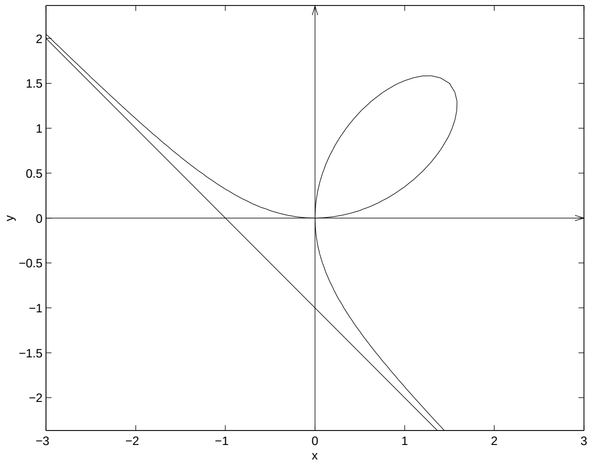

Fig. 1.1. Folium of Descartes

Example 1.1.1. Figure 1.1 shows the Folium of Descartes, introduced by R. Descartes in 1638, with its asymptotic line [227]. It can be expressed in parametric form

$$\mathbf{r}(t) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3} \right)^T, \quad -\infty < t < \infty \ \ (t \neq -1)\ , \tag{1.5}$$

where superscript T denotes transpose of a vector. For $t < -1$ the curve is located in the fourth quadrant and approaches the origin as t goes to $-\infty$. For $-1 < t < 0$ the curve is located in the second quadrant, and $t = 0$ corresponds to the origin. In the first quadrant it forms a loop moving counter-clockwise as t increases from 0 to $+\infty$. Eliminating t from (1.5), the Folium of Decartes can be also expressed in an implicit form

$$f(x, y) = x^3 + y^3 - 3xy = 0\ . \tag{1.6}$$

We can easily trace the curve using the parametric equation (1.5) by evaluating $x(t)$ and $y(t)$ for a discrete sampling of t, while such tracing is more difficult when using the implicit equation (1.6). However, determining if a point (x_0, y_0) lies on the curve is easier when using the implicit rather than the parametric equation of the curve. For example, we can verify that the point $(\frac{3}{2}, \frac{3}{2})$ lies on the curve by substituting $x = \frac{3}{2}$ and $y = \frac{3}{2}$ into implicit form and deducing that $f(\frac{3}{2}, \frac{3}{2}) = 0$. However, it is more complex to deduce this using the parametric form. We first set $x(t) = \frac{3}{2}$ which yields a cubic equation $t^3 - 2t + 1 = 0$. The roots of the cubic equation are 1, $\frac{-1\pm\sqrt{5}}{2}$. Then we substitute each root into $y(t)$ to see if it becomes equal to $\frac{3}{2}$. An alternate way to do this involves the theory of resultants from algebraic geometry that we will see in Sect. 5.4.2.

1.1.2 Space curves

The parametric representation of space curves is:

$$x = x(t), \quad y = y(t), \quad z = z(t), \quad t_1 \leq t \leq t_2 \; . \tag{1.7}$$

The implicit representation for a space curve can be expressed as an intersection curve between two implicit surfaces

$$f(x, y, z) = 0 \;\cap\; g(x, y, z) = 0 \; , \tag{1.8}$$

or parametric and implicit surfaces

$$\mathbf{r} = \mathbf{r}(u, v) \;\cap\; f(x, y, z) = 0 \; , \tag{1.9}$$

or two parametric surfaces

$$\mathbf{r} = \mathbf{p}(\sigma, t) \;\cap\; \mathbf{r} = \mathbf{q}(u, v) \; . \tag{1.10}$$

The differential geometry properties of the intersection curves between implicit surfaces are discussed in Sects. 2.2 and 2.3 as well as in Chap. 6 together with the intersection curves between parametric and implicit, and two parametric surfaces. In Sect. 5.8 algorithms for computing the intersections (1.8), (1.9) and (1.10) are discussed.

If t can be expressed as a function of x, y, or z, we can eliminate t from the parametric form (1.7) to generate the explicit form. Let us assume t is a function of x, then we have

$$y = Y(x), \quad z = Z(x) \; . \tag{1.11}$$

This is always possible at least locally when $\frac{dx}{dt} \neq 0$ [412]. Also if the two implicit equations $f(x, y, z) = 0$ and $g(x, y, z) = 0$ can be solved for two of the variables in terms of the third, for example y and z in terms of x, we obtain the explicit form (1.11). This is always possible at least locally when $\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial y} \neq 0$ [412]. Therefore the explicit equation for the space curve can be expressed as an intersection curve of two cylinders projecting the curve onto xy and xz planes.

1.2 Analytic representation of surfaces

Similar to the curve case there are mainly three ways to represent surfaces, namely parametric, implicit and explicit methods. In *parametric* representation the coordinates of a point (x, y, z) of the surface patch are expressed as functions of the parameters u and v in a closed rectangle:

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad u_1 \le u \le u_2, \quad v_1 \le v \le v_2 \; . \tag{1.12}$$

The functions $x(u, v)$, $y(u, v)$ and $z(u, v)$ are continuous and possess a sufficient number of continuous partial derivatives. The parametric surface is said to be of *class* r, if the functions have continuous (partial) derivatives up to the order r, inclusively. In case the class is not explicitly given, it is assumed that the functions have infinitely many derivatives. In vector notation the parametric surface can be specified by a vector-valued function

$$\mathbf{r} = \mathbf{r}(u, v) \; . \tag{1.13}$$

An *implicit surface* is defined as the locus of points whose coordinates (x, y, z) satisfy an equation of the form

$$f(x, y, z) = 0 \; . \tag{1.14}$$

When (1.14) is linear in variables x, y and z, it represents a plane. If (1.14) is of second degree in the variables x, y, z, it represents *quadrics* [79]

$$ax^2 + by^2 + cz^2 + dxy + eyz + hxz + kx + ly + mz + n = 0 \; . \tag{1.15}$$

Some of the quadric surfaces such as elliptic paraboloid, hyperbolic paraboloid and parabolic cylinder have explicit forms (see Fig. 8.9). Paraboloid of revolution is a special case of elliptic paraboloid where the major and minor axes are the same. The rest of the quadrics have implicit forms including ellipsoid, elliptic cone, elliptic cylinder, hyperbolic cylinder, hyperboloid of one sheet and two sheets, where the hyperboloid of revolution is a special form. The natural quadrics, sphere, circular cone and circular cylinder, which are special cases of ellipsoid, elliptic cone and elliptic cylinder, are widely used in mechanical design and CAD/CAM systems. Also they result from standard manufacturing operations such as rolling, turning, filleting, drilling and milling [149]. According to a survey conducted by the Production Automation Project group at the University of Rochester in the mid 1970's, 80-85% of mechanical parts were adequately represented by planes and cylinders, while 90-95% were modeled with the addition of cones [434, 363, 149].

If the implicit equation (1.14) can be solved for one of the variables as a function of the other two, say z is solved in terms of x and y, we obtain an explicit surface

$$z = F(x, y) \; . \tag{1.16}$$

This is always possible at least locally when $\frac{\partial f}{\partial z} \neq 0$ [166]. And if the two variables u, v of the parametric form can be solved in terms of x and y, we can substitute $u = u(x, y)$ and $v = v(x, y)$ into $z = z(u, v)$ which yields an explicit form. This is possible when $\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \neq 0$ [76]. Conversely when the explicit form $z = F(x, y)$ is given, the parametric form is derived by setting $x = u$, $y = v$, $z = F(u, v)$. Thus, the explicit form can be considered as a special case of implicit and parametric forms.

Example 1.2.1. Let us consider a hyperbolic paraboloid surface patch in the parametric form:

$$x = u + v, \quad y = u - v, \quad z = u^2 - v^2, \quad 0 \leq u, v \leq 1 \ . \tag{1.17}$$

Since we can easily solve for u and v in terms of x and y as $u = \frac{x+y}{2}$ and $v = \frac{x-y}{2}$, the explicit form is obtained as

$$z = xy, \quad 0 \leq x + y \leq 2, \ 0 \leq x - y \leq 2 \ . \tag{1.18}$$

Table 1.1. Representations of curves and surfaces

Geometry	Parametric	Implicit	Explicit
Plane curves	$x = x(t)$, $y = y(t)$ $t_1 \leq t \leq t_2$	$f(x, y) = 0$ or $\mathbf{r} = \mathbf{r}(u, v) \cap$ plane	$y = F(x)$
Space curves	$x = x(t)$, $y = y(t)$, $z = z(t)$, $t_1 \leq t \leq t_2$	$f(x, y, z) = 0 \cap g(x, y, z) = 0$ or $\mathbf{r} = \mathbf{r}(u, v) \cap f(x, y, z) = 0$ or $\mathbf{r} = \mathbf{p}(\sigma, t) \cap \mathbf{r} = \mathbf{q}(u, v)$	$y = Y(x) \cap$ $z = Z(x)$
Surfaces	$x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, $u_1 \leq u \leq u_2$, $v_1 \leq v \leq v_2$	$f(x, y, z) = 0$	$z = F(x, y)$

Table 1.1 summarizes the three representation forms for plane curves, space curves and surfaces. Table 1.2 compares the three representations [119, 116]. It is clear from the tables that the parametric form is the most versatile method among the three and the explicit is the least. Furthermore, the explicit form can always be easily converted to parametric form. Therefore we will mainly focus on the parametric and implicit forms throughout this book. Methods to fit and manipulate free-form shapes in implicit form are more complex than those for the parametric form both with respect to computation and geometric intuition. However, a considerable body of research aimed at alleviating precisely this obstacle has been published over the last fifteen years, see for example [373, 299, 16]. In this book we do not cover implicit surface fitting and design methods.

Table 1.2. Comparison of different methods of curve and surface representation

Disadvantages		
Explicit	Implicit	Parametric
• Infinite slopes are impossible if $f(x)$ is a polynomial. • Axis dependent (difficult to transform). • Closed and multivalued curves are difficult to represent.	• Difficult to fit and manipulate free form shapes. • Axis dependent. • Complex to trace.	• High flexibility complicates intersections and point classification.
Advantages		
Explicit	Implicit	Parametric
• Easy to trace.	• Closed and multivalued curves and infinite slopes can be represented. • Point classification (solid modeling, interference check) is easy. • Intersections/offsets can be represented.	• Closed and multivalued curves and infinite slopes can be represented. • Axis independent (easy to transform). • Easy to generate composite curves. • Easy to trace. • Easy in fitting and manipulating free-form shapes.